MATH 1A - MIDTERM 2

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Name: ____________________

Instructions: This midterm counts for 20% of your grade. You have 110 minutes to take this exam. **Show your steps in a clear and organized fashion, and box your answers whenever possible!** Good luck, and may the Chen Lou be with you!

1		15
2		15
3		**50**
4		20
Bonus 1		5 + 2
Bonus 2		5
Total		100

Date: Friday, July 15th, 2011.

1. *(15 points)* Using **the definition** of the derivative, calculate $f'(4)$, where:

$$f(x) = \sqrt{x}$$

2. *(15 points)* Using **the definition** of the derivative, calculate $f'(x)$, where:

$$f(x) = x^2 + x$$

3. *(50 points, 5 points each)* Find the derivatives of the following functions:

   (a) $f(x) = e^x + \cos(x) + 1$

   (b) $f(x) = x\ln(x) - x$

   (c) $f(x) = \frac{e^x}{(\sin(x))^2}$

(d) $f(x) = \sqrt{\ln(x^2 + 1)}$

(e) $f(x) = \tan(\tan(\tan(x)))$

(f) $f(x) = (\sin(x))^x$

(g) y', where $x^2 + 3xy + y^2 = 1$

(h) The equation of the tangent line to $y = x^4 + 3x$ at the point (1,4)

(i) The equation of the tangent line at $(1, 0)$ to the curve:

$$\sin(y^2 + x\sin(y)) = x^2$$

(j) $\boxed{f''(x)}$, where $f(x) = \tan^{-1}(x)$

Note: In case you don't remember the formula for the derivative of $\tan^{-1}(x)$, here's a little hint:
Let $y = \tan^{-1}(x)$, then $\tan(y) = x$, now use implicit differentiation, as well as the facts that the derivative of $\tan(x)$ is $1 + \tan^2(x)$ and $\tan(\tan^{-1}(x)) = x$.

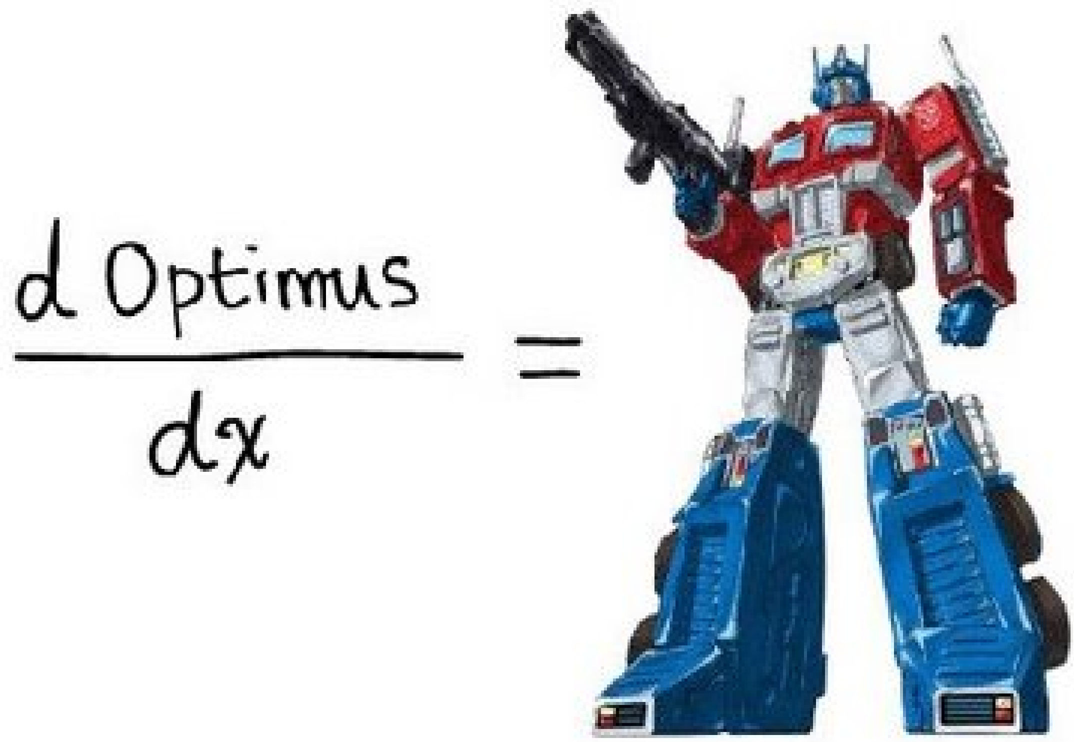
4. *(20 points)* Show that the sum of the $x-$ and $y-$ intercepts of any tangent line to the curve $\sqrt{x}+\sqrt{y}=\sqrt{c}$ is c (where c is a constant).

Hint: **At the end**, you will need the fact that:

$$x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = (\sqrt{x_0} + \sqrt{y_0})^2$$

(This page is left blank in case you need more space to work on problem 4.)

1A/Math 1A Summer/Exams/Optimus Prime.jpeg



Bonus 1 *(5 points)* Let f be a **nonzero** function which satisfies:

$$\begin{cases} f'(x) = & f(x) \\ f(0) = & 1 \end{cases}$$

Note: In particular, this implies $f'(\diamond) = f(\diamond)$ where $\diamond$ can be **anything**!

For this problem, the following property will be useful:

Property: If $g'(x) = 0$ for all x, then $g(x) = C$, where C is a constant.

(a) Show that $f(x+a) = f(x)f(a)$.

Hint: Fix a (so a is constant) and define $g(x) = \frac{f(x+a)}{f(x)}$.

(b) Show that $f(-x) = \frac{1}{f(x)}$

Hint: Define $g(x) = f(-x)f(x)$.

(c) Show that $f(ax) = f(x)^a$

Hint: Define $g(x) = \frac{f(ax)}{f(x)^a}$

Note: Do you notice something interesting going on? Remember that in class I **defined** this function f to be e^x (this was the 'awesome Peyam definition'). Just based on this definition, it wasn't obvious whether e^x is an exponential function or not. But what you've shown here is that e^x is in fact an exponential function, namely:

$$\begin{cases} e^{x+a} = & e^x e^a \\ e^{-x} = & \frac{1}{e^x} \\ e^{ax} = & (e^x)^a \end{cases}$$

And this follows just from the fact that e^x is a function which is its own derivative!!!

How cool is that? :)

(d) *(2 extra points)* In fact, the 'reverse' statement is true too! Namely, if f is a function with:

$$\begin{cases} f(a+b) = & f(a)f(b) \\ f(0) = & 1 \\ f'(0) = & 1 \end{cases}$$

Show that $f(x) = e^x$.

Hint: All you need to show is that $f'(x) = f(x)$ for all x.

Bonus 2 *(5 points)* The following bonus problem is meant to show you that derivatives can behave in very strange ways!

Note: See the comments on page 18 for an interesting discussion of this problem!

(a) Find an example of a function f with $\lim_{x \to \infty} f(x) = 0$ but $\lim_{x \to \infty} f'(x)$ does not exist. Prove that your answer is correct.

Hint: $f(x) = \frac{\sin(x)}{x}$ is **not** an example, because although it goes to 0, it does not oscillate wildly enough. How can you modify f to make the oscillations worse?

(b) Find an example of a function f that is differentiable at 0, but whose derivative is not continuous at 0. Prove that your answer is correct.

Hint: The answer is $f(x) = x^N \sin(\frac{1}{x})$, where N is an integer which you'll have to choose.

Hint: To show that the derivative is not continuous at 0, first calculate $f'(0)$ using the definition of the derivative, then calculate $f'(x)$ using differentiation rules. If f' were continuous, then we would have $\lim_{x\to 0} f'(x) = f'(0)$. Show that this is bogus!

Note: See the discussion on the next page!

Discussion of Bonus 2:

Note: Part (a) has a nice physical interpretation: it says that if you're driving your car towards a certain point your velocity might not necessarily go to 0, even if you drive for a long time. All is not lost, though! It can be shown that if $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} f'(x)$ **exists**, then in fact $\lim_{x\to\infty} f'(x) = 0$ as well!

Note: In order to avoid strange examples as in Part (b), mathematicians usually require functions not only to be differentiable, but also **continuously differentiable**, i.e. f' must be continuous as well.

We say f is C^1 if f is once differentiable and f' is continuous, f is C^2 if f is twice differentiable and f'' is continuous, etc. We have the following inclusions:

$$C^0 \supseteq C^1 \supseteq C^2 \supseteq \cdots \supseteq C^\infty \supseteq C^\omega$$

where $\supseteq$ means 'includes', and C^0 is the set of continuous functions, C^∞ is the set of infinitely differentiable functions, and C^ω is the set of analytic functions, i.e. the set of functions which have a power series expansion at every point (essentially infinite polynomials, see Math 1B).

The interesting fact is that for **complex** functions (i.e. $f(z)$ where z is a complex number such as i or $1+i$), we do not have such a distinction, i.e. $C^0 = C^1 = C^2 \cdots = C^\infty = C^\omega$.

In particular, a complex function that is once differentiable is infinitely differentiable! WOW!!!

(Scrap work)

Any comments about this exam? (too long? too hard?)